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Stochastic resonance and improvement by noise in optimal detection strategies

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Abstract

A stochastic resonance effect, under the form of a noise-improved performance, is shown feasible for a whole range of optimal detection strategies, including Bayesian, minimum error-probability, Neyman–Pearson, and minimax detectors. In each case, situations are demonstrated where the performance of the optimal detector can be improved (locally) by raising the level of the noise. This is obtained with a nonlinear signal-noise mixture where a non-Gaussian noise acts on the phase of a periodic signal.

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1. Introduction

Stochastic resonance is a phenomenon in which some processing done on a signal can be improved by the action of the noise [1–3]. The feasibility of stochastic resonance has now been reported in a large variety of processes, under many different forms [4–9]. Yet, until very recently, stochastic resonance as an improvement of the performance by noise, was limited to suboptimal devices or processors. In the context of detection problems, various aspects of stochastic resonance have been investigated [10–18], yet with improvement by noise limited to suboptimal detection strategies. Very recently [19,20], stochastic resonance has been shown feasible also in optimal processing, in a Bayesian detection problem

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on a nonlinear signal-noise mixture with non-Gaussian noise. In the present paper we consider the same type of detection problem, and we extend the demonstration of feasibility of stochastic resonance, to a whole range of standard optimal detection strategies, in a coherent perspective.

2. Strategies for optimal detection

In this section, we briefly review standard strategies for optimal detection. Our aim here is to exhibit the definition of the optimal detectors we will be considering, and the way their performance is assessed. Classical proofs and developments can be found in [21,22]. Later on, our point will be to show that situations can be found where the performance of each one of these optimal detectors can be improved by operating them at higher noise levels, over some ranges of the noise.

In a standard detection problem, one among two known signals $s_0(t)$ or $s_1(t)$ may be mixed to a noise $\eta(t)$, the resulting mixture forming the observable signal $x(t)$. Observation of $x(t)$ at N distinct times t_k , for $k = 1$ to N , provides N data points $x_k = x(t_k)$. From the data $\mathbf{x} = (x_1, \dots, x_N)$, it is to be decided whether $x(t)$ is formed by $\eta(t)$ mixed to $s_0(t)$ (hypothesis H_0) or to $s_1(t)$ (hypothesis H_1). Any conceivable detector is equivalent to a partition of \mathbb{R}^N into two disjoint complementary subsets \mathcal{R}_0 and \mathcal{R}_1 , such that when \mathbf{x} falls in \mathcal{R}_i then the detector decides H_i , for $i \in \{0, 1\}$.

2.1. Bayesian detection

If the prior probabilities are known, P_0 for hypothesis H_0 , and $P_1 = 1 - P_0$ for H_1 , it is possible to assess the performance of a given detector by means of a Bayesian cost. One introduces four elementary costs C_{ij} of deciding H_i when H_j holds, $i, j \in \{0, 1\}$, with necessarily $C_{10} > C_{00}$ and $C_{01} > C_{11}$ to penalize wrong decisions. The average Bayesian cost is then

$$C = P_0 C_{00} \int_{\mathcal{R}_0} p(\mathbf{x}|H_0) d\mathbf{x} + P_1 C_{01} \int_{\mathcal{R}_0} p(\mathbf{x}|H_1) d\mathbf{x} \\ + P_0 C_{10} \int_{\mathcal{R}_1} p(\mathbf{x}|H_0) d\mathbf{x} + P_1 C_{11} \int_{\mathcal{R}_1} p(\mathbf{x}|H_1) d\mathbf{x}, \quad (1)$$

where $p(\mathbf{x}|H_i)$ is the probability density for observing \mathbf{x} when H_i holds, and $\int d\mathbf{x}$ stands for the N -dimensional integral $\int \dots \int dx_1 \dots dx_N$. The cost C of Eq. (1) is minimized by the optimal Bayesian detector that uses the likelihood ratio

$$L(\mathbf{x}) = \frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} = \frac{\Pr\{\mathbf{x}|H_1\}}{\Pr\{\mathbf{x}|H_0\}} \quad (2)$$

to implement the test

$$L(\mathbf{x}) \underset{H_0}{\overset{H_1}{\geq}} \frac{P_0 C_{10} - C_{00}}{P_1 C_{01} - C_{11}}, \quad (3)$$

and by doing so achieves the minimal cost

$$C_{\min} = \frac{1}{2} [P_1(C_{01} + C_{11}) + P_0(C_{10} + C_{00})] - \frac{1}{2} \int_{\mathbb{R}^N} |P_1(C_{01} - C_{11})p(\mathbf{x}|\mathbf{H}_1) - P_0(C_{10} - C_{00})p(\mathbf{x}|\mathbf{H}_0)| d\mathbf{x}. \quad (4)$$

2.2. Minimum error-probability detection

In the special case where $C_{00} = C_{11} = 0$ and $C_{10} = C_{01} = 1$, the cost C of Eq. (1) represents the overall probability of detection error P_{er} . The optimal Bayesian detector of Eq. (4) then represents the detector minimizing P_{er} , also known as the maximum a posteriori probability (MAP) detector since test (4) becomes equivalent to a test comparing $\Pr\{\mathbf{H}_1|\mathbf{x}\}/\Pr\{\mathbf{H}_0|\mathbf{x}\}$ to 1.

2.3. Neyman–Pearson detection

When P_0 and P_1 are unknown, a strategy to implement an optimal detection is to seek to maximize the probability of detection

$$P_d = \int_{\mathcal{R}_1} p(\mathbf{x}|\mathbf{H}_1) d\mathbf{x}, \quad (5)$$

while keeping the probability of false alarm

$$P_f = \int_{\mathcal{R}_1} p(\mathbf{x}|\mathbf{H}_0) d\mathbf{x} \quad (6)$$

no larger than a prescribed level $P_{f,\text{sup}}$.

This constrained maximization is achieved by the optimal Neyman–Pearson detector, which also implements a likelihood-ratio test

$$\begin{array}{c} \mathbf{H}_1 \\ L(\mathbf{x}) \geq \mu(P_{f,\text{sup}}), \\ \mathbf{H}_0 \end{array} \quad (7)$$

with a threshold $\mu(P_{f,\text{sup}})$, a function of $P_{f,\text{sup}}$, which is found from Eq. (6) by imposing $P_f \leq P_{f,\text{sup}}$, with the subset \mathcal{R}_1 defined as those points \mathbf{x} for which $p(\mathbf{x}|\mathbf{H}_1) > \mu p(\mathbf{x}|\mathbf{H}_0)$.

2.4. Minimax detection

In the absence of known P_0 and P_1 , another strategy for an optimal detection, which does not require to specify a $P_{f,\text{sup}}$, is to look for the value P_0^* of P_0 that maximizes the minimal cost C_{\min} of Eq. (4). The optimal minimax detector then implements the likelihood-ratio test of Eq. (4) with P_0 set to P_0^* . For any detector, the Bayesian cost C in Eq. (1) is a function of P_0 , which goes through a maximum for some $P_0 \in [0, 1]$. This

maximum turns out to be minimized by the optimal minimax detector¹ defined above, and the minimal value achieved for this maximum is expressible as

$$\begin{aligned} C_{\text{minimax}} &= C_{10} + (C_{00} - C_{10}) \int_{\mathcal{R}_0} p(\mathbf{x}|\mathbf{H}_0) d\mathbf{x} \\ &= C_{01} + (C_{11} - C_{01}) \int_{\mathcal{R}_1} p(\mathbf{x}|\mathbf{H}_1) d\mathbf{x}. \end{aligned} \quad (8)$$

So for an unknown P_0 , adopting the optimal minimax detector minimizes the highest cost we would incur if nature were to impose us the least favorable P_0 .

3. Detection with phase noise

We now apply the strategies of Section 2 to a specific detection problem. This is the same detection problem that was used in [19,20] for the first report of a noise-improved performance in an optimal Bayesian detector, and that we adopt here to test a whole family of optimal detectors. We consider a periodic “mother” wave $w(t)$ of period unity. A possibility could be $w(t) = \sin(2\pi t)$, but $w(t)$ will be further specified later. One of the two signals to be detected is the wave $w(t)$ with frequency ν_0 , i.e., $s_0(t) = w(\nu_0 t)$; the other signal is the same wave $w(t)$ with frequency $\nu_1 \neq \nu_0$, i.e., $s_1(t) = w(\nu_1 t)$. A noise $\eta(t)$ acts on signals $s_0(t)$ and $s_1(t)$ as a phase noise, so as to form the observable signal

$$x(t) = w[\nu_0 t + \eta(t)] \quad (\text{hypothesis } \mathbf{H}_0) \quad \text{or} \quad (9)$$

$$x(t) = w[\nu_1 t + \eta(t)] \quad (\text{hypothesis } \mathbf{H}_1). \quad (10)$$

Such periodic signals corrupted by a phase noise arise, for instance, when a periodic wave propagates in a fluctuating medium or traverses a fluctuating interface. Phase noise is present in oscillators, phase-locked loops, coherent imaging. A simple concretization of the present setting is realized by a plane wave radiated or received by a transducer subjected to a random motion producing the phase noise.

Based on the data $\mathbf{x} = (x_1, \dots, x_N)$ it is to be decided whether the wave corrupted by the phase noise has frequency ν_0 or ν_1 . In order to allow a complete analytical treatment of the optimal detection strategies presented in Section 2, we consider, as in [19,20], the case where $w(t)$ is a square wave of period 1 with $w(t) = 1$ when $t \in [0, 1/2)$ and $w(t) = -1$ when $t \in [1/2, 1)$. In such a case, the possible values of the observations x_k reduce to ± 1 , and quantities such $p(\mathbf{x}|\mathbf{H}_i) d\mathbf{x}$ define the finite probabilities $\Pr\{\mathbf{x}|\mathbf{H}_i\}$ which are nonzero only at the locations of the 2^N states $\mathbf{x} = (\pm 1, \dots, \pm 1)$.

We assume the noise samples $\eta(t_k)$ statistically independent for distinct t_k 's, so that the conditional probabilities factorize as $\Pr\{\mathbf{x}|\mathbf{H}_i\} = \prod_{k=1}^N \Pr\{x_k|\mathbf{H}_i\}$. Also, the samples $\eta(t_k)$ are identically distributed, with cumulative distribution function $F_\eta(u)$ and probabil-

¹ Also, it can be shown that the Bayesian cost C in Eq. (1) attached to the optimal minimax detector turns out to be equal to the constant C_{minimax} of Eq. (8) for any P_0 .

ity density function $f_\eta(u) = dF_\eta/du$. We then have

$$\Pr\{x_k = 1|\mathbf{H}_1\} = \Pr\{w[v_1 t_k + \eta(t_k)] = 1\}, \quad (11)$$

$$= \Pr\left\{v_1 t_k + \eta(t_k) \in \bigcup_{\ell} [\ell, \ell + 1/2)\right\}, \quad (12)$$

$$= \Pr\left\{\eta(t_k) \in \bigcup_{\ell} [\ell - v_1 t_k, \ell - v_1 t_k + 1/2)\right\}, \quad (13)$$

$$= \sum_{\ell=-\infty}^{+\infty} \int_{\ell - v_1 t_k}^{\ell - v_1 t_k + 1/2} f_\eta(u) du, \quad (14)$$

$$= \sum_{\ell=-\infty}^{+\infty} [F_\eta(\ell - v_1 t_k + 1/2) - F_\eta(\ell - v_1 t_k)], \quad (15)$$

ℓ integer, and

$$\Pr\{x_k = -1|\mathbf{H}_1\} = 1 - \Pr\{x_k = 1|\mathbf{H}_1\}. \quad (16)$$

In the same way, we have

$$\Pr\{x_k = 1|\mathbf{H}_0\} = \sum_{\ell=-\infty}^{+\infty} [F_\eta(\ell - v_0 t_k + 1/2) - F_\eta(\ell - v_0 t_k)] \quad (17)$$

and

$$\Pr\{x_k = -1|\mathbf{H}_0\} = 1 - \Pr\{x_k = 1|\mathbf{H}_0\}. \quad (18)$$

The probabilities $\Pr\{\mathbf{x}|\mathbf{H}_i\}$ that follow from Eqs. (15)–(18), allow an explicit implementation of the optimal detectors of Section 2 along with the assessment of their performance, as a function of the properties of the noise conveyed by $F_\eta(u)$.

4. Improvement by noise

The optimal Bayesian detector of Eq. (4) achieves the minimal cost C_{\min} of Eq. (4) which here is computable as

$$\begin{aligned} C_{\min} = & \frac{1}{2} [P_1(C_{01} + C_{11}) + P_0(C_{10} + C_{00})] \\ & - \frac{1}{2} \sum_{x_1 \in \{-1, 1\}} \dots \sum_{x_N \in \{-1, 1\}} |P_1(C_{01} - C_{11}) \Pr\{x_1|v_1\} \dots \Pr\{x_N|v_1\} \\ & - P_0(C_{10} - C_{00}) \Pr\{x_1|v_0\} \dots \Pr\{x_N|v_0\}|, \end{aligned} \quad (19)$$

the multiple sum running over the 2^N states accessible to the data \mathbf{x} .

For illustration of the possibility of a stochastic resonance, Fig. 1 shows the performance of the optimal Bayesian detector measured by the cost C_{\min} of Eq. (19), as a function of

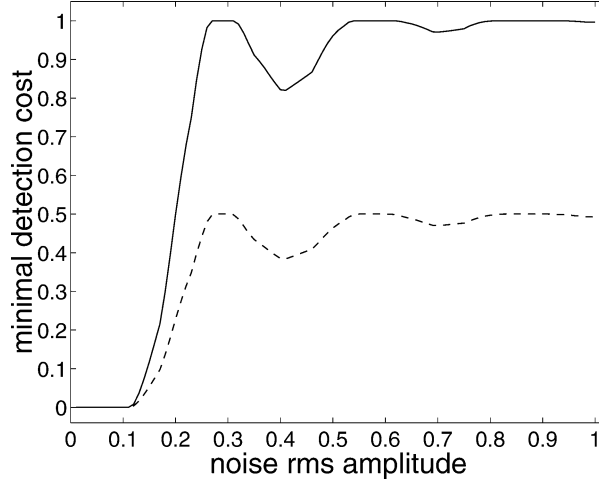


Fig. 1. Minimal detection cost C_{\min} from Eq. (19) of the optimal Bayesian detector, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$. Also $C_{00} = C_{11} = 0$, $P_0 = 0.5$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 13$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_{13} = 2.4$. Solid line: $C_{10} = 2$ and $C_{01} = 5$, dashed line: $C_{10} = 1$ and $C_{01} = 2$.

the rms amplitude σ_η of the phase noise $\eta(t)$ which has been chosen zero-mean uniform. In the representative conditions of Fig. 1, the evolutions of the cost C_{\min} are clearly non-monotonic as the noise level σ_η increases. This demonstrates the possibility, over some ranges of the noise level σ_η , of reducing the cost of the optimal detection by operating at higher noise levels. It is this very possibility that we want to emphasize, and that we interpret as a form of stochastic resonance, the nonmonotonic evolution of the optimal performance, when the noise level increases, instead of a monotonic degradation as commonly observed in standard optimal detection processes.

For another illustration, we choose for the phase noise $\eta(t)$, the family of Gaussian-mixture densities $f_\eta(u) = f_{\text{gm}}(u/\sigma_\eta)/\sigma_\eta$ defined through the standardized probability density

$$f_{\text{gm}}(u) = \frac{1}{2\sqrt{2\pi}\sqrt{1-m^2}} \left\{ \exp\left[-\frac{(u+m)^2}{2(1-m^2)}\right] + \exp\left[-\frac{(u-m)^2}{2(1-m^2)}\right] \right\}, \quad (20)$$

with $0 < m < 1$, and cumulative distribution function

$$F_{\text{gm}}(u) = \frac{1}{2} + \frac{1}{4} \left[\operatorname{erf}\left(\frac{u+m}{\sqrt{2}\sqrt{1-m^2}}\right) + \operatorname{erf}\left(\frac{u-m}{\sqrt{2}\sqrt{1-m^2}}\right) \right]. \quad (21)$$

Figure 2 shows the performance of the optimal Bayesian detector measured by the cost C_{\min} of Eq. (19), as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$, in two representative sets of conditions. Again, the evolutions of Fig. 2 are clearly nonmonotonic as the noise level σ_η increases, demonstrating the same type of stochastic resonance with the optimal Bayesian detector. Figure 2 also presents numerical valida-

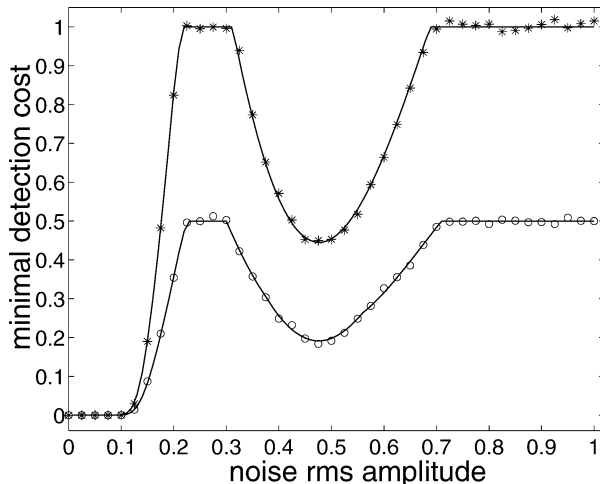


Fig. 2. Minimal detection cost C_{\min} of the optimal Bayesian detector, as a function of the rms amplitude σ_{η} of the Gaussian-mixture noise $\eta(t)$ with $m = 0.95$. The solid lines are C_{\min} of Eq. (19); the discrete points are C_{\min} numerically evaluated with 10^4 Monte Carlo trials of the test (4) for each σ_{η} ; with $C_{10} = 2$ and $C_{01} = 5$ (*), $C_{10} = 1$ and $C_{01} = 2$ (o). Also $C_{00} = C_{11} = 0$, $P_0 = 0.5$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 6$ data samples equispaced with time step 0.3 from $t_1 = 0$ to $t_6 = 1.5$.

tions of the theoretical performance, through a Monte Carlo implementation of the optimal detector of Eq. (4).

The same possibility of improvement by noise also exists for the performance of the optimal MAP detector, or minimum error-probability detector, measured by the (minimum) probability of detection error P_{er} . Figures 3 and 4 illustrate the possibility of reducing P_{er} , by increasing the level of noise σ_{η} , over some ranges, successively for the zero-mean uniform noise and for the Gaussian-mixture noise from Eq. (20).

For the optimal Neyman–Pearson detector, with the nonlinear signal-noise mixture with phase noise, we did not find it possible to obtain a general explicit analytical characterization of the threshold $\mu(P_{\text{f,sup}})$ of Eq. (8), for an arbitrary noise density $f_{\eta}(u)$. Nevertheless here, for a reasonable number of data points N leading to $2^N = N_S$ different states accessible to the data \mathbf{x} , it is feasible to exhaustively test all the 2^{N_S} possible partitions $(\mathcal{R}_0, \mathcal{R}_1)$, select those associated to a P_{f} in Eq. (6) no larger than $P_{\text{f,sup}}$, and among them retain the one with maximal P_{d} in Eq. (5). This was done so at each σ_{η} in various configurations, and the results are shown in Figs. 5 and 6, successively for the zero-mean uniform noise and for the Gaussian-mixture noise from Eq. (20). The evolutions of Figs. 5 and 6 demonstrate, for the optimal Neyman–Pearson detector, the possibility of increasing its probability of detection P_{d} , by raising the level of noise σ_{η} , over some ranges. This is again a nonmonotonic evolution of the performance of the optimal detector when the noise level is raised, that we interpret as a form of stochastic resonance.

Finally, it is possible to find P_0^* that maximizes C_{\min} of Eq. (19), opening the way to the optimal minimax detector and its performance measured by the cost C_{minimax} of Eq. (8). Evolutions of C_{minimax} are shown in Figs. 7 and 8, successively for the zero-mean uniform noise and for the Gaussian-mixture noise from Eq. (20). The results of Figs. 7

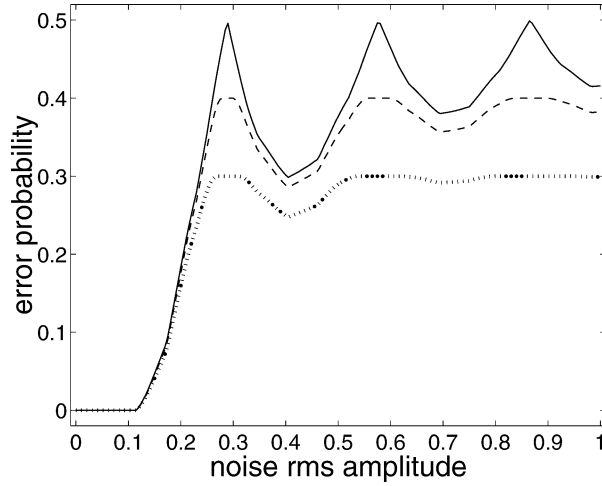


Fig. 3. Probability of detection error P_{er} from Eq. (19) of the optimal MAP detector, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$. Also $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 11$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_{11} = 2$. Solid line: $P_0 = 0.5$, dashed line: $P_0 = 0.4$, dotted line: $P_0 = 0.3$.

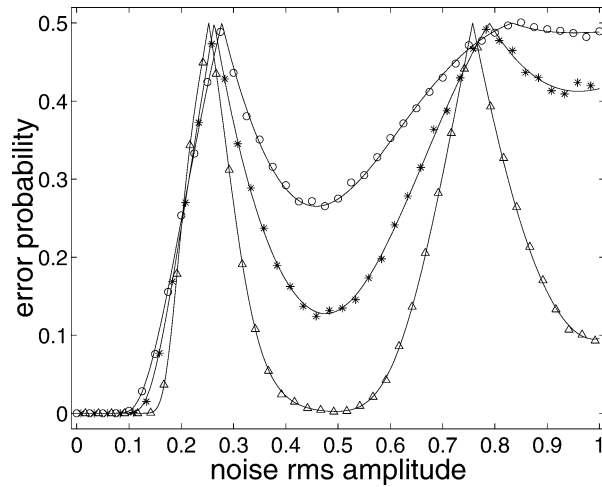


Fig. 4. Probability of detection error P_{er} of the optimal MAP detector, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$. The solid lines are P_{er} from Eq. (19); the discrete points are P_{er} numerically evaluated from 10^4 Monte Carlo trials of the MAP test (4) for each σ_η ; with $m = 0.9$ (\circ), $m = 0.95$ ($*$), $m = 0.99$ (\triangle). Also $P_1 = 0.5$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 6$ data samples equispaced with time step 0.3 from $t_1 = 0$ to $t_6 = 1.5$.

and 8 establish the possibility of decreasing the cost C_{\minimax} , by increasing the level of noise σ_η , over some ranges. This again reveals the same type of nonmonotonic evolution of the performance of the optimal detector when the noise level is raised, instead of a monotonic degradation.

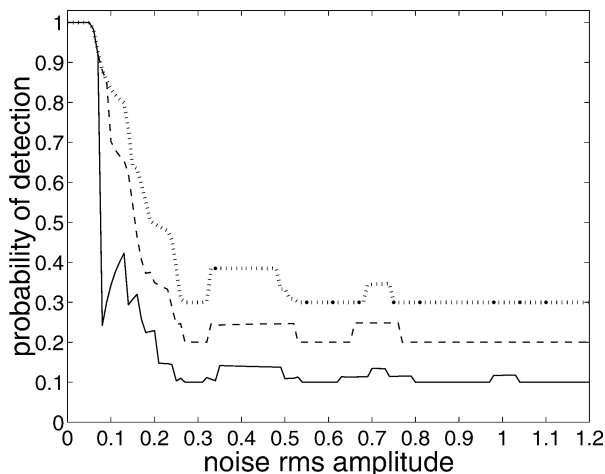


Fig. 5. Probability of detection P_d of Eq. (5) of the optimal Neyman–Pearson detector, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$, with $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 3$ data samples measured at $t_1 = 0.1$, $t_2 = 0.4$, and $t_3 = 0.6$. Also $P_{f,\text{sup}} = 0.1$ (solid line), $P_{f,\text{sup}} = 0.2$ (dashed line), $P_{f,\text{sup}} = 0.3$ (dotted line).

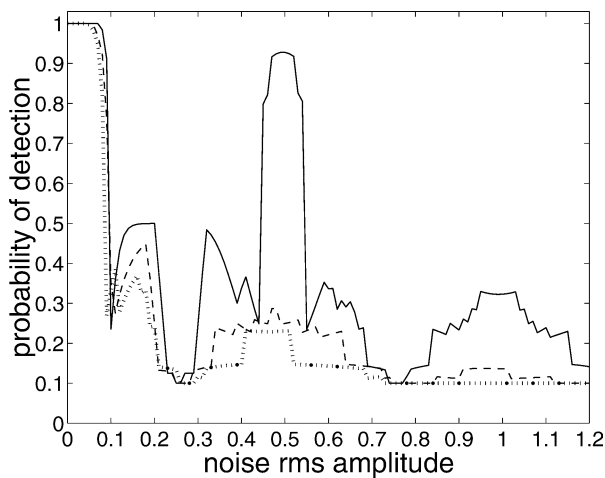


Fig. 6. Probability of detection P_d of Eq. (5) of the optimal Neyman–Pearson detector, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ with $m = 0.9$ (dotted line), $m = 0.95$ (dashed line), $m = 0.99$ (solid line). Also $P_{f,\text{sup}} = 0.1$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 3$ data samples measured at $t_1 = 0.1$, $t_2 = 0.4$, and $t_3 = 0.6$.

5. Discussion

The present results essentially stand for a proof of feasibility in principle, by direct examination, of a stochastic resonance effect under the form of a noise-improved performance in a whole range of standard optimal detection strategies. Detailed analyses of the

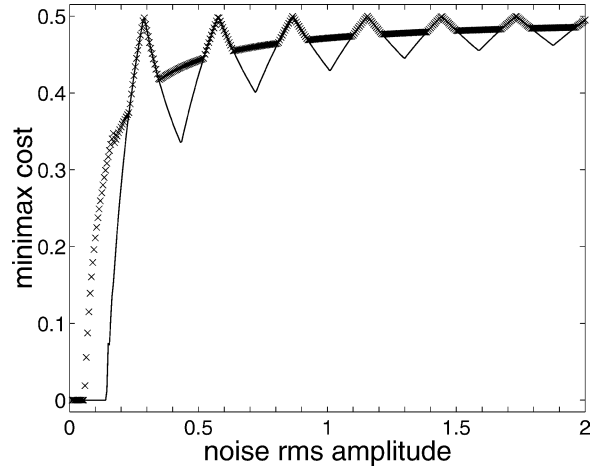


Fig. 7. Minimax cost C_{minimax} of Eq. (8) of the optimal minimax detector, as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$. Also $C_{00} = C_{11} = 0$, $C_{01} = C_{10} = 1$. Solid line: $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 6$ data samples equispaced with time step 0.25 from $t_1 = 0$ to $t_6 = 1.5$. Crosses (\times): $\nu_0 = 1$, $\nu_1 = 1/3$, with $N = 4$ data samples equispaced with time step 0.6 from $t_1 = 0$ to $t_4 = 1.8$.

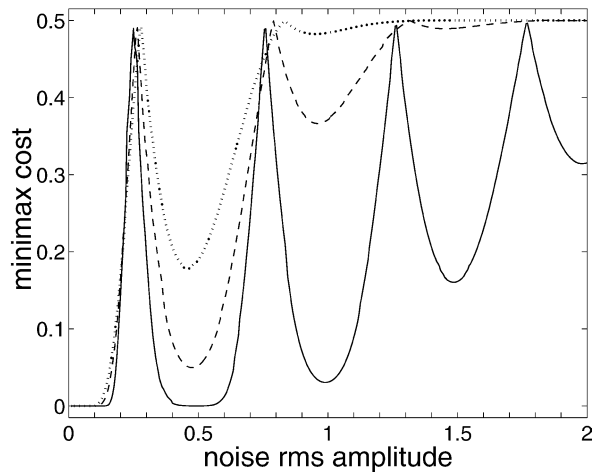


Fig. 8. Minimax cost C_{minimax} of Eq. (8) of the optimal minimax detector, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ with $m = 0.9$ (dotted line), $m = 0.95$ (dashed line), $m = 0.99$ (solid line). Also $C_{00} = C_{11} = 0$, $C_{01} = C_{10} = 1$, $\nu_0 = 1$, $\nu_1 = 2/3$, with $N = 11$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_{11} = 2$.

influence of the distinct parameters playing a role in the detection process are not accomplished in the present study, and remain open for subsequent investigations.

Among the important parameters is the waveform of the periodic input the frequency of which we seek to detect. We have chosen here a square wave, essentially because this case is simple enough to lend itself to a complete analytical treatment, which was at the same

time verified by Monte Carlo simulations of the optimal detectors, as shown in Figs. 2 and 4. These conditions allowed our proof of feasibility in principle, with the double check of an analytical treatment and a numerical simulation. The case of detection on a sine wave, for instance, is a more complicated one, that remains open for future study.

It is to note that in all the optimal detection strategies we have tested, the performance is always at its best when the detectors are operated with no noise, at $\sigma_\eta = 0$. This is a common behavior which can reasonably be expected from any optimal detector, to achieve its absolute best performance when no noise is present. Then, when a small amount of noise is added above $\sigma_\eta = 0$, the performance of the optimal detectors naturally starts to degrade compared to the absolute best performance at $\sigma_\eta = 0$. Our main finding is that this evolution of the performance of the optimal detectors does not continue as a monotonic degradation: at higher levels of noise, ranges of σ_η exist where the performance improves, at least locally, when the noise is further raised. A pre-existing amount of noise has to be present, in order to have access to improvement by noise via the stochastic resonance effect. But a pre-existing noise is usually the rule in a detection problem, compared to the situation of no noise.

A qualitative explanation can be proposed for the observed effect of improvement by noise in optimal detection. For the observable signal-noise mixture $x(t)$ of Eqs. (9)–(10), let us assume that the corrupting phase noise $\eta(t)$ is a binary or dichotomous noise, randomly switching between the two levels $-\sigma_\eta$ and $+\sigma_\eta$. When $\sigma_\eta = 1$, because of the periodicity 1 of the wave $w(t)$, the binary phase noise $\eta(t)$ has actually no action on the signal $x(t)$, which is preserved exactly as if no noise were present. Therefore, at the noise level $\sigma_\eta = 1$, the performance of any detector will return to the performance of this detector at $\sigma_\eta = 0$. It can be expected that the performance will first degrade as σ_η is raised above zero, but later on, the performance will improve as σ_η approaches one, to return to its initial value at $\sigma_\eta = 0$. A similar effect can be expected when the binary phase noise $\eta(t)$ switches between $\pm\sigma_\eta = \pm 0.5$, or any integer multiple of such configurations. The Gaussian-mixture noise from Eq. (20) that we have tested in Figs. 2, 4, 6, and 8, precisely tends to the binary noise at $\pm\sigma_\eta$ when $m \rightarrow 1$, and it reflects this qualitative behavior we have explained for the binary noise. In this respect, our present report can be seen as a quantitative analysis, extended to any type of noise, of the qualitative behavior anticipated for the binary noise. The important and nonobvious point is that the behavior anticipated for the binary noise and which leads to a nonmonotonic evolution of the performance, is in actuality preserved for non-binary noises, noises with a bimodal structure like the Gaussian-mixture noise from Eq. (20), but also unimodal noises like the uniform noise tested in Figs. 1, 3, 5, and 7. The nonmonotonic behavior of the performance is essentially caused by the action of the nonlinearity itself, as in Eqs. (9)–(10), which mixes the information signal and the noise, through a phase coupling which introduces a sensitivity to the amplitude of the noise in relation to the periodicity² of $w(t)$. The noise, somehow,

² It is to be noted that the level of the phase noise $\eta(t)$ ought not be measured modulo 1. The noise level is given, in principle at an arbitrary value, prior to the nonlinear coupling with the information signal through Eqs. (9)–(10). It is for instance the arbitrary level at which the transducer evoked at the end of the paragraph after Eq. (10), is randomly shaken. Subsequently, it is the action of the nonlinear coupling via Eqs. (9)–(10), which brings the (arbitrary) amplitude of the noise to have an action on the phase between 0 and 1.

is able to bring some phase shift which places the nonlinear coupling in a better configuration for the detection task, by making the waves at ν_0 and at ν_1 more distinguishable. And this mechanism can take place for various types of noise, not necessarily binary noise, not necessarily bimodal noise, especially unimodal noise. The outcome, which is the main point of this report, is the possibility of (local) improvement of the performance of *optimal* detectors as the noise level increases.

In the tested configurations, with a square wave $w(t)$, the stochastic resonance effect was not observed with Gaussian noise ($m = 0$), but it appears with Gaussian-mixture noise, and in a more pronounced way as we depart more from the Gaussian, with $m \rightarrow 1$ (evolution to binary noise), and also with uniform noise. The uniform noise is a generalized Gaussian noise [14] with exponent $+\infty$, while the standard Gaussian noise corresponds to an exponent of 2. We have observed, with a few other instances of generalized Gaussian noises, that the stochastic resonance effect tends to vanish smoothly (the nonmonotonic evolution of the performance gradually gives way to a monotonic degradation) as the exponent tends to 2 from above. The effect disappears for an exponent of 2 or below. Further studies will be useful for a more detailed appreciation of the influence of the type of the noise on the stochastic resonance. Many other settings and conditions have been reported where a form of stochastic resonance (in suboptimal processing) takes place with Gaussian noise [1–3,11,23]. In the future, conditions may well be found allowing stochastic resonance in optimal processing with Gaussian noise. This issue remains open for future investigation. In this perspective, studies are currently under way to investigate the possibility of extending stochastic resonance to optimal estimation [24].

Important nonlinear processes that have been shown to lend themselves to many interesting forms of stochastic resonance are the neural processes [4,13,25–31]. Neurons are intrinsically nonlinear devices, they naturally operate in noisy conditions (of external or internal origins), and they achieve high efficiency for signal and information processing. In addition, neurons handle the signals essentially under the form of trains of stereotyped pulses (action potentials) mostly invariant in shape, but with their coding capability supported by the temporal sequencing of the pulses along the trains. This can be viewed as a coding in phase, and the conditions of the present study, with phase coupling and phase noise, could carry some relevance for neural information processing, this perspective also remaining open for detailed investigation.

The present results contribute a new step in the inventory and analysis of the properties and potentialities of stochastic resonance, at large, understood as an effect of noise-improved information processing. Beyond the present proof of feasibility in principle, of a stochastic resonance in optimal detection strategies, further studies can proceed in many directions to extend our knowledge of noise-enhanced information processing.

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