

NOISE IMPROVEMENTS IN STOCHASTIC RESONANCE: FROM SIGNAL AMPLIFICATION TO OPTIMAL DETECTION

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Received (24 April 2002)

Revised ()

Accepted (20 September 2002)

It is demonstrated that benefits from the noise can be gained at various levels in stochastic resonance. Raising the noise can produce signal amplification as well as signal-to-noise ratio improvement, input-output gain exceeding unity in signal-to-noise ratio, and enhanced performance in optimal processing. This series of benefits is successively exhibited in the processing of a periodic signal coupled to a white noise through essentially static nonlinearities. Especially, it is established that noise benefits in stochastic resonance can extend up to optimal processing, by considering an optimal Bayesian detector whose performance is improvable by raising the level of the noise.

Keywords: Stochastic resonance; signal; noise; optimal detection.

1. Introduction

Since its introduction some twenty years ago in the context of climate dynamics [1], the phenomenon of stochastic resonance has experienced large varieties of extensions, developments and observations in many areas of natural sciences (for overviews, see for instance [2–4]). The common feature preserved among all these manifestations of stochastic resonance, is the possibility of improving the transmission or the processing of a signal by means of an increase in the level of the noise coupled to this signal. It is possible to summarize many forms observed for stochastic resonance by means of the scheme of Fig. 1.

Stochastic resonance, as illustrated by Fig. 1, involves four essential ingredients:

- (i) an information signal $s(t)$, which can be of many different types, deterministic, periodic or non, random;
- (ii) a noise $\eta(t)$, whose statistical properties can be of various kinds: white or colored, Gaussian or non;
- (iii) a transmission or processing system, which generally is nonlinear, receiving $s(t)$ and $\eta(t)$ as inputs under the influence of which it produces the output signal $y(t)$;

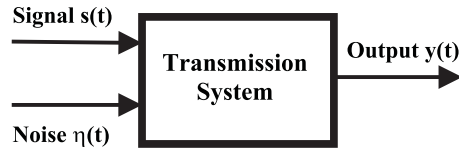


Fig 1. A general scheme for stochastic resonance

(iv) a measure of performance, which quantifies the efficacy of the processing or transmission of $s(t)$ into $y(t)$ in the presence of $\eta(t)$, and which can also be of many different types, according to the context: signal-to-noise ratio, correlation coefficient, Shannon mutual information, . . .

Stochastic resonance then takes place each time it is possible to improve the measure of performance by means of an increase in the level of the noise $\eta(t)$.

For a relatively long period of time after its introduction, stochastic resonance has essentially been addressed with a sinusoidal $s(t)$ added to a white Gaussian $\eta(t)$ transmitted by a nonlinear dynamic system governed by a double-well potential and measured by a signal-to-noise ratio in the frequency domain [5]. It is relatively recently that stochastic resonance has “exploded” in many other directions through variations and extensions over the four basic ingredients (i)–(iv). Stochastic resonance, as a signal-aided-by-noise effect, has thus been reported with nonperiodic and with random $s(t)$ [6, 7], with non-Gaussian [4, 8] and with colored [9,10] $\eta(t)$, with single-well [11] and with static nonlinearities [12], measured by correlation functions [6] and by information-theoretic quantities [13–15] and many other performance indices. At the same time, stochastic resonance has received concrete materializations in many specific areas of natural sciences, such as electronic circuits [16, 17], optical devices [18, 19], biophysical [20], psychophysical [21] or neural processes [22, 23], chemical reactions [24, 25], material-science phenomena [26, 27], . . . All these results progressively accumulating now establish stochastic resonance as a general paradigm for complex and nonlinear noisy processes, endowed with many different specific forms and numerous domains of applicability.

In the present paper, we propose a specific perspective on stochastic resonance, based on a grading or ranking of the various benefits that can be gained from the noise. Each step in the process will illustrate a conceptual advance in the nature of the benefit afforded by the noise. Each of these advances is a priori counterintuitive in some way, up to a point that historically the mere possibility of most of these steps has remained uncertain for some time in the development of stochastic resonance. Explicitly, we shall successively show that raising the noise can produce signal amplification as well as signal-to-noise ratio improvement, input-output gain exceeding unity in signal-to-noise ratio, and enhanced performance in optimal processing. This series of benefits will be exhibited in the scheme of Fig. 1, with a periodic signal $s(t)$ and a stationary white noise $\eta(t)$ of cumulative distribution function $F_\eta(u)$ and probability density function $f_\eta(u) = dF_\eta/du$, the system realizing the signal-noise coupling essentially being of a static or memoryless nature.

2. Amplification of Signal Amplitude

Amplification of the amplitude of a spectral line contributed by a small periodic signal buried in noise, has been shown possible in stochastic resonance in double-well dynamic systems [5, 28]. Here, we demonstrate this possibility in static nonlinearities, which are simple enough to lend themselves to an exact theoretical analysis where both the shape of the periodic waveform and the probability density of the white noise are arbitrary, a type of analysis that is not available exactly in more complex dynamic nonlinearities.

When $s(t)$ is a deterministic periodic signal with period T_s , the system of Fig. 1 elicits an output $y(t)$ which generically is a random signal with a frequency spectrum formed by a broadband continuous noise background (contributed by $\eta(t)$), out of which emerge spectral lines at integer multiples of $1/T_s$ (contributed by $s(t)$). The influence of $s(t)$ in $y(t)$ can be quantified by the magnitude of the spectral line at $1/T_s$ in the output spectrum.

When [12] the nonlinear system of Fig. 1 is a static or memoryless nonlinearity $g(\cdot)$ realizing

$$y(t) = g[s(t) + \eta(t)] , \quad (1)$$

the magnitude of the spectral line at harmonics n/T_s in the output spectrum is $|\bar{Y}_n|$, with

$$\bar{Y}_n = \frac{1}{T_s} \int_0^{T_s} E[y(t)] \exp\left(-in\frac{2\pi}{T_s}t\right) dt , \quad (2)$$

and the expectation $E[y(t)]$ at a fixed time t computable as

$$E[y(t)] = \int_{-\infty}^{+\infty} g(u) f_\eta[u - s(t)] du . \quad (3)$$

When $g(\cdot)$ realizes a two-level quantizer

$$y(t) = \text{sign}[s(t) + \eta(t) - \theta] = \pm 1 \quad (4)$$

with threshold θ , one gets

$$E[y(t)] = 1 - 2F_\eta[\theta - s(t)] . \quad (5)$$

For the transmission of a sine wave $s(t) = \sin(2\pi t/T_s)$, the resulting output signal amplitude $|\bar{Y}_1|$ is shown in Fig. 2A. When the amplitude of $s(t)$ is below the quantization threshold θ , no transition is induced in $y(t)$ in the absence of noise. As the input noise level σ_η is raised above zero, a cooperative effect can take place where the noise $\eta(t)$ assists the periodic input $s(t)$ in overcoming the threshold. A periodic influence begins to be discernable in $y(t)$. This is quantified by the output amplitude $|\bar{Y}_1|$ which increases from zero as σ_η is raised, and $|\bar{Y}_1|$ culminates at a maximum for an optimal nonzero noise level, as visible in Fig. 2A. When $\sigma_\eta \rightarrow +\infty$, we have $E[y(t)] \rightarrow 0$ and $|\bar{Y}_1|$ returns to zero.

Figure 2A thus illustrates an amplification of the periodic component in the noisy output $y(t)$ which is realized by the operation of the noise, a form of stochastic resonance.

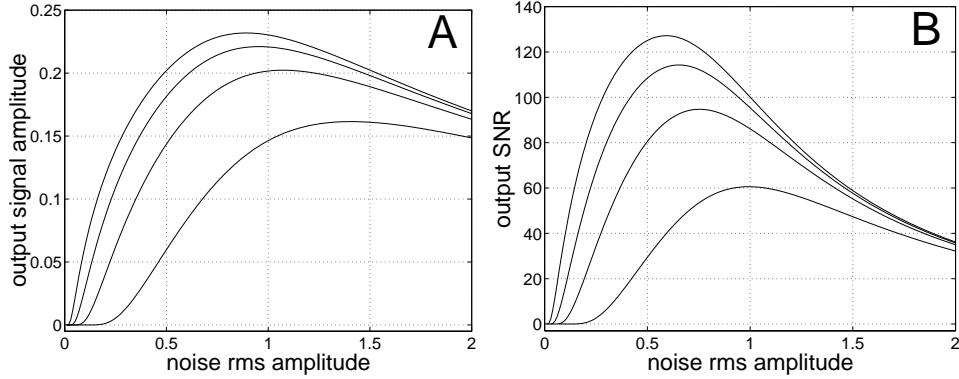


Fig 2. Transmission of $s(t) = \sin(2\pi t/T_s)$ by Eq. (4) with, from the upper to the lowest curve of each panel, $\theta = 1.05, 1.1, 1.2, 1.5$. In abscissa is the rms amplitude σ_η of the input noise $\eta(t)$ chosen zero-mean Gaussian. Panel A: Output signal amplitude $|\bar{Y}_1|$ from Eq. (2). Panel B: Output signal-to-noise ratio $\mathcal{R}_{\text{out}}(1/T_s)$ from Eq. (6) with $\Delta t \Delta B = 10^{-3}$.

3. Signal-to-Noise Ratio Improvement

Increase in the input noise that increases the magnitude of the periodic component in the output may at the same time increase the output noise background out of which the periodic line emerges. A further characterization [5] defines an output signal-to-noise ratio $\mathcal{R}_{\text{out}}(n/T_s)$ in the frequency domain, as the power contained in the coherent spectral line at n/T_s divided by the power contained in the noise background in a small frequency band ΔB around n/T_s .

When [12] the nonlinear system of Fig. 1 is the static nonlinearity $g(\cdot)$ of Eq. (1), this output signal-to-noise ratio $\mathcal{R}_{\text{out}}(n/T_s)$ is expressible as

$$\mathcal{R}_{\text{out}}\left(\frac{n}{T_s}\right) = \frac{|\bar{Y}_n|^2}{\overline{\text{var}(y)} \Delta t \Delta B}, \quad (6)$$

with

$$\overline{\text{var}(y)} = \frac{1}{T_s} \int_0^{T_s} \text{var}[y(t)] dt \quad (7)$$

and

$$\text{var}[y(t)] = \int_{-\infty}^{+\infty} g^2(u) f_\eta[u - s(t)] du - \left(\int_{-\infty}^{+\infty} g(u) f_\eta[u - s(t)] du \right)^2, \quad (8)$$

also Δt is the time resolution of the measurement (i.e. the signal sampling step in a discrete time implementation).

In the case of the transmission by the two-level quantizer of Eq. (4), one gets

$$\text{var}[y(t)] = 4F_\eta[\theta - s(t)] \left\{ 1 - F_\eta[\theta - s(t)] \right\}. \quad (9)$$

The resulting output signal-to-noise ratio $\mathcal{R}_{\text{out}}(1/T_s)$ from Eq. (6) is shown in Fig. 2B in the same conditions as in Fig. 2A for the transmission of $s(t) = \sin(2\pi t/T_s)$.

The evolutions in Fig. 2B illustrate another aspect of the noise-assisted transmission, with a signal-to-noise ratio (SNR) this time which can be improved by addition of noise, another form of stochastic resonance.

4. Input–Output SNR Gain

If the signal-plus-noise mixture at the input of the stochastic resonator is available, a further benefit that can be sought concerns the possibility of obtaining an output SNR larger than the input SNR in stochastic resonance. This possibility has remained uncertain for a long time. The issue was touched in [29,30] to result in no positive answer. Proofs were given in [31,32] that, in the limit of a small periodic signal with Gaussian noise, the output SNR cannot exceed the input SNR. Later, by circumventing the small-signal limit or the Gaussian noise condition, examples were explicitly exhibited of an SNR larger at the output than at the input in stochastic resonance, by simulation in an excitable system [33], or analytically in static nonlinearities [34].

With static nonlinearities as in Eq. (1), the input SNR $\mathcal{R}_{\text{in}}(1/T_s)$ for the harmonic n/T_s is expressible as

$$\mathcal{R}_{\text{in}}\left(\frac{n}{T_s}\right) = \frac{|S_n|^2}{\sigma_\eta^2 \Delta t \Delta B}, \quad (10)$$

with

$$S_n = \frac{1}{T_s} \int_0^{T_s} s(t) \exp\left(-in\frac{2\pi}{T_s}t\right) dt. \quad (11)$$

The input–output SNR gain $G_{\text{SNR}}(n/T_s)$ in the transmission then follows as

$$G_{\text{SNR}}\left(\frac{n}{T_s}\right) = \frac{\mathcal{R}_{\text{out}}(n/T_s)}{\mathcal{R}_{\text{in}}(n/T_s)} = \frac{|\overline{Y}_n|^2 / \overline{\text{var}}(y)}{|S_n|^2 / \sigma_\eta^2}. \quad (12)$$

We consider the case where the nonlinearity $g(\cdot)$ of Eq. (1) is a hard threshold realizing

$$y(t) = \begin{cases} 0 & \text{if } s(t) + \eta(t) \leq \theta, \\ 1 & \text{if } s(t) + \eta(t) > \theta. \end{cases} \quad (13)$$

In this case, Eqs. (3) and (8) lead respectively to

$$\text{E}[y(t)] = 1 - F_\eta[\theta - s(t)], \quad (14)$$

and

$$\text{var}[y(t)] = F_\eta[\theta - s(t)] \left\{ 1 - F_\eta[\theta - s(t)] \right\}. \quad (15)$$

The T_s -periodic input $s(t)$ is taken as a train of rectangular pulses with duration T , i.e.

$$s(t) = \begin{cases} 1 & \text{for } t \in [0, T), \\ 0 & \text{for } t \in [T, T_s), \end{cases} \quad (16)$$

These conditions can be seen as mimicking, in a crude way, the transmission of spike trains by a neuron.

Figure 3A shows the input-output SNR gain $G_{\text{SNR}}(1/T_s)$ from Eq. (12) for the transmission of the periodic pulse train with Gaussian noise. SNR gains culminating at values much larger than unity are clearly accessible as the level of the input noise is raised.

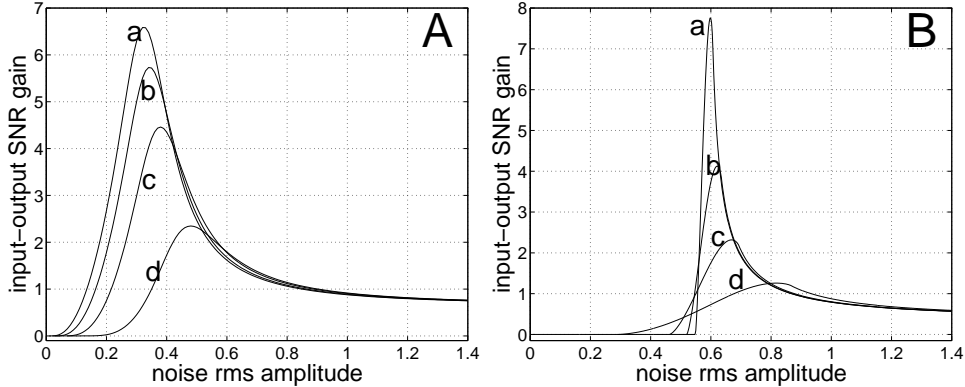


Fig 3. Input-output SNR gain $G_{\text{SNR}}(1/T_s)$ from Eq. (12) as a function of the rms amplitude σ_η of the zero-mean input noise $\eta(t)$. Panel A: $\eta(t)$ Gaussian, for the pulse train $s(t)$ of Eq. (16) with $T = 0.01T_s$, transmitted by the nonlinearity of Eq. (13) with threshold $\theta = 1.05$ (a), 1.1 (b), 1.2 (c), 1.5 (d). Panel B: $\eta(t)$ uniform, for the sinusoid $s(t) = A \sin(2\pi t/T_s)$, transmitted by the nonlinearity of Eq. (17) with threshold $\theta = 1$ and $A = 0.05$ (a), 0.1 (b), 0.2 (c), 0.5 (d).

It is also possible to obtain an input-output SNR gain larger than unity in the transmission of a sine wave $s(t) = A \sin(2\pi t/T_s)$. We consider the case where the nonlinearity $g(\cdot)$ of Eq. (1) is a three-level symmetric quantizer realizing

$$y(t) = \begin{cases} -1 & \text{if } s(t) + \eta(t) \leq -\theta, \\ 0 & \text{if } -\theta < s(t) + \eta(t) < \theta, \\ 1 & \text{if } s(t) + \eta(t) \geq \theta. \end{cases} \quad (17)$$

In this case, Eqs. (3) and (8) lead respectively to

$$E[y(t)] = 1 - F_\eta[\theta - s(t)] - F_\eta[-\theta - s(t)], \quad (18)$$

and

$$\begin{aligned} \text{var}[y(t)] &= \left\{1 - F_\eta[\theta - s(t)]\right\} F_\eta[\theta - s(t)] + \left\{1 - F_\eta[-\theta - s(t)]\right\} F_\eta[-\theta - s(t)] \\ &\quad + 2 \left\{1 - F_\eta[\theta - s(t)]\right\} F_\eta[-\theta - s(t)]. \end{aligned} \quad (19)$$

Figure 3B shows the SNR gain $G_{\text{SNR}}(1/T_s)$ from Eq. (12) for the transmission of the sine wave with uniform noise. Again, SNR gains culminating at values much larger than unity are clearly accessible in stochastic resonance. It is to note that with these simple static nonlinearities, we have not been able to observe SNR gains exceeding unity simultaneously with a sine wave and Gaussian noise, although this is quite possible with a sine wave and non-Gaussian noise, or with a periodic nonsinusoidal wave and Gaussian noise, as illustrated by Fig. 3.

The evolutions of Fig. 3 clearly demonstrate the possibility of raising the input–output SNR gain well above unity by addition of noise, another form of stochastic resonance.

5. Optimal Detection Improved by Noise

So far, we have considered a signal $s(t)$ in the presence of the noise $\eta(t)$ and processed by a nonlinear system according to the scheme of Fig. 1. In such conditions, various measures of performance (output signal amplitude, output SNR, input–output SNR gain) were shown improvable by an increase in the level of the input noise. This, we interpreted as various forms of stochastic resonance. Yet, so far, the systems whose performances have been improved by the noise were suboptimal devices. For instance, with the systems of Eq. (4) or Eq. (13) or Eq. (17), with a given input $s(t)$ at a given level of noise, one can seek to optimize the threshold θ so as to obtain the optimal system that maximizes a specified measure of performance (output SNR, input–output SNR gain) [35]. For such an optimal system then, it can be verified that the (maximal) performance monotonically degrades as the noise level is raised. In other words, noise improvement of the performance only occurs with suboptimal devices, with an ill-positioned threshold upon which no control is available, and where noise addition brings assistance to the signal in overcoming this non-optimal threshold.

Nevertheless it is possible in other conditions to observe a stochastic resonance effect, under the form of a noise-improved performance, in optimal devices, as we shall now show. To demonstrate this possibility, we consider a nonlinear signal-noise mixture where the noise $\eta(t)$ acts on the phase of the periodic signal $s(t)$. For a detection task on this noisy signal, a standard Bayesian approach is implemented. The optimal Bayesian detector achieving the minimum C_{\min} of an average detection cost is derived. Conditions where C_{\min} is reduced when the noise level is raised will be exhibited, establishing for an optimal detector a new instance of stochastic resonance.

The signal $s(t)$ is a periodic wave which can have either the frequency ν_0 or the frequency $\nu_1 \neq \nu_0$, i.e. $s(t) = w(\nu_0 t)$ with a priori probability P_0 , or $s(t) = w(\nu_1 t)$ with a priori probability $P_1 = 1 - P_0$, where $w(t)$ describes a mother waveform with period unity, for example $w(t) = \sin(2\pi t)$.

$s(t)$ is then corrupted by a phase noise $\eta(t)$ so as to yield the observable signal

$$y(t) = w[\nu_0 t + \eta(t)] \quad (\text{hypothesis } H_0), \text{ or} \quad (20)$$

$$y(t) = w[\nu_1 t + \eta(t)] \quad (\text{hypothesis } H_1). \quad (21)$$

A concretization of such $y(t)$ is provided by a plane wave impinging on a transducer subjected to a random motion producing the phase noise. N data points $y_k = y(t_k)$ are collected on $y(t)$ at N distinct times t_k for $k = 1$ to N . Based on $\mathbf{y} = (y_1, \dots, y_N)$ it is to be decided between hypothesis H_0 or H_1 , i.e. whether $s(t)$ corrupted by the phase noise has frequency ν_0 or ν_1 . Other forms of stochastic resonance with phase noise have been considered in [36, 37].

A given detector will decide H_0 whenever the observation \mathbf{y} falls in the region \mathcal{R}_0 of \mathbb{R}^N , and it will decide H_1 whenever \mathbf{y} falls in the complementary region \mathcal{R}_1 of \mathbb{R}^N . One specifies four elementary costs C_{ij} of performing decision H_i when

hypothesis H_j holds true, i, j from 0 to 1, with necessarily $C_{10} > C_{00}$ and $C_{01} > C_{11}$ to penalize wrong decisions. The average detection cost is then

$$C = P_0 C_{00} \int_{\mathcal{R}_0} p(\mathbf{y}|H_0) d\mathbf{y} + P_1 C_{01} \int_{\mathcal{R}_0} p(\mathbf{y}|H_1) d\mathbf{y} + P_0 C_{10} \int_{\mathcal{R}_1} p(\mathbf{y}|H_0) d\mathbf{y} + P_1 C_{11} \int_{\mathcal{R}_1} p(\mathbf{y}|H_1) d\mathbf{y}, \quad (22)$$

where $p(\mathbf{y}|H_j)$ is the probability density for observing \mathbf{y} when H_j holds true, and $\int \cdot d\mathbf{y}$ stands for the N -dimensional integral $\int \dots \int \cdot dy_1 \dots dy_N$.

Since \mathcal{R}_0 and \mathcal{R}_1 are complementary in \mathbb{R}^N , one has

$$\int_{\mathcal{R}_0} p(\mathbf{y}|H_j) d\mathbf{y} = 1 - \int_{\mathcal{R}_1} p(\mathbf{y}|H_j) d\mathbf{y}, \quad (23)$$

which, inserted in Eq. (22), yields

$$C = P_0 C_{00} + P_1 C_{01} + \int_{\mathcal{R}_1} \left[P_0 (C_{10} - C_{00}) p(\mathbf{y}|H_0) - P_1 (C_{01} - C_{11}) p(\mathbf{y}|H_1) \right] d\mathbf{y}. \quad (24)$$

The optimal Bayesian detector reaches the overall minimum of the average detection cost C by making as negative as possible the integral in Eq. (24), i.e. by including into \mathcal{R}_1 all and only those points \mathbf{y} for which its integrand is negative. This amounts to using the likelihood ratio

$$L(\mathbf{y}) = \frac{p(\mathbf{y}|H_1)}{p(\mathbf{y}|H_0)} = \frac{\Pr\{\mathbf{y}|H_1\}}{\Pr\{\mathbf{y}|H_0\}} \quad (25)$$

to implement the test

$$L(\mathbf{y}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0}{P_1} \frac{C_{10} - C_{00}}{C_{01} - C_{11}}. \quad (26)$$

The minimum C_{\min} reached for the average cost C by the optimal Bayesian detector of Eq. (26) then comes out as

$$C_{\min} = \int_{\mathbb{R}^N} \min \left[P_0 C_{00} p(\mathbf{y}|H_0) + P_1 C_{01} p(\mathbf{y}|H_1), P_0 C_{10} p(\mathbf{y}|H_0) + P_1 C_{11} p(\mathbf{y}|H_1) \right] d\mathbf{y}. \quad (27)$$

Since $\min(a, b) = (a + b - |a - b|)/2$, one has

$$C_{\min} = \frac{1}{2} \left[P_1 (C_{01} + C_{11}) + P_0 (C_{10} + C_{00}) \right] - \frac{1}{2} \int_{\mathbb{R}^N} \left| P_1 (C_{01} - C_{11}) p(\mathbf{y}|H_1) - P_0 (C_{10} - C_{00}) p(\mathbf{y}|H_0) \right| d\mathbf{y}. \quad (28)$$

In order to apply this general optimal Bayesian detection scheme to our specific detection problem stated by Eqs. (20)–(21), we consider $\eta(t)$ to be a white noise, so that the conditional probabilities factorize as $\Pr\{\mathbf{y}|\nu_1\} = \prod_{k=1}^N \Pr\{y_k|\nu_1\}$ and $\Pr\{\mathbf{y}|\nu_0\} = \prod_{k=1}^N \Pr\{y_k|\nu_0\}$. To allow a complete analytical treatment of the optimal detector, we further consider the simple case where $w(t)$ is a square wave of period unity with $w(t) = 1$ when $t \in [0, 1/2)$ and $w(t) = -1$ when $t \in [1/2, 1)$. We then have the probabilities

$$\Pr\{y_k = 1|\nu_1\} = \Pr\{w[\nu_1 t_k + \eta(t_k)] = 1\} \quad (29)$$

$$= \Pr\left\{\nu_1 t_k + \eta(t_k) \in \bigcup_{\ell} [\ell, \ell + 1/2)\right\} \quad (30)$$

$$= \Pr\left\{\eta(t_k) \in \bigcup_{\ell} [\ell - \nu_1 t_k, \ell - \nu_1 t_k + 1/2)\right\} \quad (31)$$

$$= \sum_{\ell=-\infty}^{+\infty} \int_{\ell - \nu_1 t_k}^{\ell - \nu_1 t_k + 1/2} f_{\eta}(u) du \quad (32)$$

$$= \sum_{\ell=-\infty}^{+\infty} [F_{\eta}(\ell - \nu_1 t_k + 1/2) - F_{\eta}(\ell - \nu_1 t_k)], \quad (33)$$

ℓ integer, and

$$\Pr\{y_k = -1|\nu_1\} = 1 - \Pr\{y_k = 1|\nu_1\}. \quad (34)$$

In the same way, we have

$$\Pr\{y_k = 1|\nu_0\} = \sum_{\ell=-\infty}^{+\infty} [F_{\eta}(\ell - \nu_0 t_k + 1/2) - F_{\eta}(\ell - \nu_0 t_k)], \quad (35)$$

and

$$\Pr\{y_k = -1|\nu_0\} = 1 - \Pr\{y_k = 1|\nu_0\}. \quad (36)$$

Given $F_{\eta}(u)$, P_0 and the costs C_{ij} , when a realization of \mathbf{y} is observed, Eqs. (33)–(36) allow an explicit evaluation of the likelihood ratio of Eq. (25) under the form $L(\mathbf{y}) = (\prod_{k=1}^N \Pr\{y_k|\nu_1\}) / (\prod_{k=1}^N \Pr\{y_k|\nu_0\})$, making possible an explicit implementation of the optimal Bayesian detector of Eq. (26). This optimal detector achieves the minimal detection cost C_{\min} which, according to Eq. (28), is computable as

$$\begin{aligned} C_{\min} &= \frac{1}{2} [P_1(C_{01} + C_{11}) + P_0(C_{10} + C_{00})] - \\ &\frac{1}{2} \sum_{y_1=-1}^1 \dots \sum_{y_N=-1}^1 \left| P_1(C_{01} - C_{11}) \Pr\{y_1|\nu_1\} \dots \Pr\{y_N|\nu_1\} - \right. \\ &\left. P_0(C_{10} - C_{00}) \Pr\{y_1|\nu_0\} \dots \Pr\{y_N|\nu_0\} \right|, \end{aligned} \quad (37)$$

the multiple sum running over the 2^N possible states for the data \mathbf{y} .

We consider for the phase noise, the class of zero-mean Gaussian mixture with standardized probability density ($0 < m < 1$)

$$f_{\text{gm}}(u) = \frac{1}{2\sqrt{2\pi}\sqrt{1-m^2}} \left\{ \exp\left[-\frac{(u+m)^2}{2(1-m^2)}\right] + \exp\left[-\frac{(u-m)^2}{2(1-m^2)}\right] \right\}. \quad (38)$$

With $f_{\eta}(u) = f_{\text{gm}}(u/\sigma_{\eta})/\sigma_{\eta}$, Fig. 4 shows different nonmonotonic evolutions of the performance C_{min} of the optimal Bayesian detector, as the noise rms amplitude σ_{η} grows. The conditions of Fig. 4A, with $C_{00} = C_{11} = 0$ and $C_{10} = C_{01} = 1$, allow one to interpret C_{min} of Eq. (37) directly as the overall probability of detection error P_{er} of the optimal Bayesian detector.

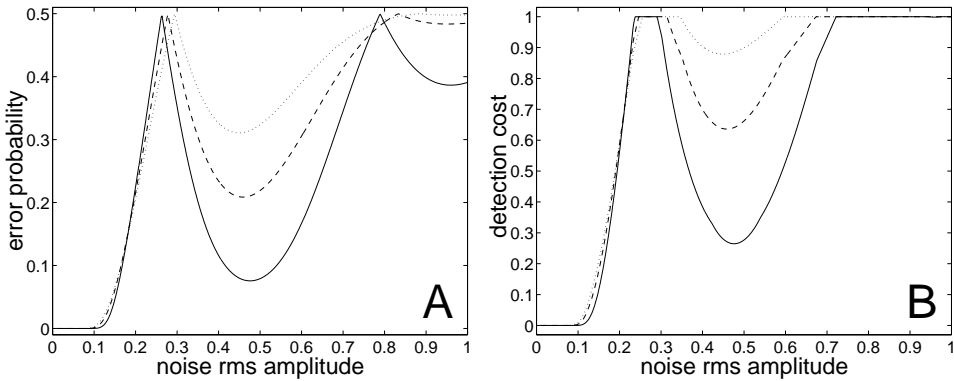


Fig 4. Minimum detection cost C_{min} of Eq. (37) for the optimal detector of Eq. (26), as a function of the rms amplitude σ_{η} of the Gaussian-mixture noise $\eta(t)$ from Eq. (38) with $m = 0.85$ (dotted line), $m = 0.9$ (dashed line), $m = 0.95$ (solid line). Also $P_0 = 0.5$, $\nu_0 = 1$, $\nu_1 = 2/3$, $N = 8$ data samples equispaced with time step 0.3 from $t_1 = 0$ to $t_8 = 2.1$. Panel A: $C_{00} = C_{11} = 0$, $C_{10} = C_{01} = 1$, then the cost C_{min} is interpretable as the overall probability of detection error P_{er} . Panel B: $C_{00} = C_{11} = 0$, $C_{10} = 2$ and $C_{01} = 5$.

Figure 4 reveals nonmonotonic evolutions of the measure of performance C_{min} of the optimal detector, as the level of the Gaussian-mixture noise is raised. It is to note that in this setting, Gaussian noise would lead to a monotonic degradation (increase) of C_{min} , but many non-Gaussian noises, like uniform, Gaussian-mixture, dichotomous noises yield nonmonotonic evolutions of C_{min} . Therefore, Fig. 4 clearly demonstrates the possibility of improving the performance achieved by the optimal Bayesian detector, by means of an increase of the noise level σ_{η} , over some ranges of σ_{η} . Of course, for the optimal detection of Fig. 4, the cost or the probability of error is always zero at zero noise. This is a reasonable behavior that can be expected for any optimal detection scheme. But if a nonzero amount of native noise pre-exists, associated for the optimal detector to a given optimal performance C_{min} (the best performance among all detectors operating at this level of noise), then Fig. 4 establishes that it can sometimes be preferable to bring the detector to work at a larger noise level so as to benefit from a better optimal performance C_{min} . In other words, the performance of the optimal detector can be improved by operating at higher noise levels, over some ranges of the noise. This is another form of stochastic resonance, as a noise-improved performance in optimal detection.

6. Conclusion

We have demonstrated that stochastic resonance allows one to obtain from the noise, benefits of different levels, culminating with the possibility of noise-enhanced performance in optimal processing. The present approach has considered a periodic signal, whose processing receives improvement from the noise. Yet a similar picture could be obtained for noise-aided processing of nonperiodic or random signals, quantified by other appropriate measures of performance. Forms of noise improvement occurring in optimal processing as in Section 5, are relatively new in the inventory of the properties of stochastic resonance [38, 39]. In this direction, other forms of optimal processing enhanced by noise can certainly be identified and analyzed. This contributes to extend the potentialities afforded by stochastic resonance, and beyond by nonlinear processes, for signal and information processing in the presence of noise.

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